Random-Variate Generation

A random variate is a variable generated from uniformly distributed pseudorandom numbers. Depending on how they are generated, a random variate can be uniformly or nonuniformly distributed. Random variates are frequently used as the input to simulation models

(Neelamkavil 1987, p. 119).

Random Variate

- In simulation, it's often necessary to generate random observations from probability distributions in order to determine future event times. Such a generated observation is referred to as a random variate.
- Assume that a distribution has been specified.
- Some widely-used techniques for generating random variates.
  - Inverse transform
  - Convolution
  - Acceptance-rejection
  - Composition (not discussed)

Inverse Transform Technique

- Used to sample from
  - Exponential
  - Uniform
  - Weibull
  - Triangular
  - Empirical
  - Discrete distributions
- Generate U ~ U (0, 1)
- Return $X = F^{-1}(U)$.
  - F inverse denotes the solution of the equation r = F(x) in terms of r, not 1/F.
  - Most useful when the cdf, F(X), is of simple form that its inverse can be easily computed.
- R, R_1, R_2, … represent numbers uniformly distributed on (0, 1)

Inverse Transform – Exponential Distribution

- Goal: develop a procedure for generating values X_1, X_2, … which have an exponential distribution.
- Steps:
  - Compute the cdf of the desired random variable X.
    - For exponential, the cdf is $F(x) = 1 - e^{-\lambda t}$, x = 0
  - Set F(X) = R on the range of X.
  - Solve the equation F(X) = R for X in terms of R.
    - For exponential, see (8.1): random-variate generator for the exponential distrituion.
  - Generate uniform random numbers, R_1, R_2, … and compute the desired random variates, see (8.3)
- Example 8.1

Inverse Transform – Exponential Distribution

- Why does the random variables X_i generated by this procedure have exponential distribution?
  - See figure 8.2
  - Compute the cumulative probability for x_0
    - $P(X_1 = x_0) = P(R_1 = F(x_0)) = F(x_0)$
    - Since $0 = F(x_0) = 1$, $P(R_1 = F(x_0)) = F(x_0)$ shows that R_i is uniformly distributed.

Inverse Transform – Other Continuous Distributions

- Same steps can be applied to uniform, Weibull, and triangle distributions.
- Empirical continuous distributions:
  - No known theoretical distributions can be found.
  - Interpolate between the observed data points to fill in the gaps.
    - First sort the data points in increasing order.
    - Assign a probability to each interval.
    - Draw the observed empirical cdf, F(x) hat and compute the slope of the *i*th line segment.
    - Calculate $F^{-1}(R)$.
  - What if the sample size is large?
    - Summarize the data into a frequency distribution with a much smaller # of intervals and fit the empirical cdf to the frequency distribution.
    - Use (8.11)
    - Relative short intervals are recommended to give more accurate underlying cdf.
    - Trade-off between accuracy of the estimating cdf and computational efficiency.
- Continuous distributions without a closed-form inverse:
  - Examples include normal, gamma, and beta
  - Approximate the inverse cdf.

Inverse Transform – Discrete Distributions

- Use the inverse transform technique either numerically or algebraically.
- Other techniques are sometimes used for certain distributions, such as the convolution technique for the binomial distribution.
- Empirical distributions
  - Table-lookup approach: example 8.4.
    - Interpolation is not required.
  - Algebraic approach: example 8.5
- The geometric distribution: example 8.7
  - Compare with equation (8.1)

Transformation for Normal Distribution

- Given X ~ N(0, 1), we can obtain X' ~ N(μ, s²) by setting X' = μ + sX. So, we restrict attention to generating standard normal random variates.
- A commonly used method for generating N(0, 1) random variates was developed by Box and Muller. There are faster algorithms, but this method maintains a 1-1 correspondence between the random numbers used and the N(0, 1) random variates produced.

- Generate Z_1 and Z_2 as IID U(0, 1) from two independent random variables R_1 and R_2, then set
  - $Z_1 = (-2 \ln R_1)^{1/2} \cos (2pR_2)$
  - $Z_2 = (-2 \ln R_1)^{1/2} \sin (2pR_2)$
  - Z_1 and Z_2 are IID N(0, 1) random variates.

- This method gives the desired random variates in pairs.
- Note: this method is valid in principle, i.e., if R_1 and R_2 are truly independent. There is a problem if R_1 and R_2 are two adjacent random numbers produced by a linear congruential generator. In this case, Z_1 and Z_2 are not truly independently normally distributed.

Convolution Method

- Applies when the random variable X can be expressed as a sum of other random variables that are identically independent distributed and easier to generate than X
  - $X = Y_1 + Y_2 + \ldots + Y_m$
  - X has distribution function F and Y_i has distribution function G.

1. Generate m Y_i IID each with distribution function G.
2. Return $X = Y_1 + Y_2 + \ldots + Y_m$

- Erlang RV X (K, ?): sum of K independent exponential random variables, X_i (i = 1, 2, …K), each having mean 1/K?.
  - $X = S^K X_i$, where $X_i = -1/K?\ \ln R_i$
  - $X = -1/K?\ \ln (?^K R_i)$, used when K is small
  - Example 8.9

Acceptance-Rejection Technique

- Previous methods are more direct, while A-R is less direct. Used when the direct methods fail or are inefficient.
- Basic idea and steps:
  1. Generate a random number R.
  2. Test a condition based on R and some computation derived for the desired distribution.
  3. If the condition is satisfied, return X computed from a formula → accept.

     otherwise → reject, go to step 1 and try again.
- Different distributions have different steps and formulas.
  - Poisson distribution: examples 8.10 & 8.11
  - Gamma distribution: example 8.12

Generating Continuous Random Variates

- Uniform: inverse-transform
- Exponential: inverse-transform
- Erlang: convolution
- Gamma: acceptance-rejection
- Weibull: inverse-transform
- Normal: direct transformation, see section 8.2
- Triangular: inverse-transform
- Empirical: inverse-transform

Generating Discrete Random Variates

- Bernoulli: inverse-transform
- Uniform: inverse-transform
- Arbitrary discrete: inverse-transform
- Binomial: convolution
- Geometric: inverse-transform
- Negative binomial: convolution
- Poisson: acceptance-rejection